Insights into Solvability by Radicals

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Abstract

For a given polynomial arbitrary equation in integers, there is no guarantee that it is solvable in radicals. If the degree is less than five then there are well-known formulas for giving the roots in terms of radicals and the coefficients of the original equation. The fact that polynomials above degree four have no solution did not come easy, as it takes many important conceptual steps to realize that methods of solving equations of lower degree can not generalize further. Many of these ideas at the time of their inception seemed impossible and were met with great opposition. Even so, through the work of Abel, Galois and their predecessors, the conditions for solvability in radicals is now known precisely. The fundamental steps and turning points in this endeavor is the concern of this paper.

A Brief History of Solvability

From the very first use of the word *algebra*, it has been associated with solving equations, generally through the operations of addition, subtraction, multiplication, division and extraction of roots. But from within the rules and procedures of algebra there are some equations which cannot be solved using these operations alone. What would al-Khwarizmı have said about such solutions? Would he have contended that they did not exist, or that they represented something unnatural? Even Cardano rejected negative solutions, while Descartes called them “less than nothing”. Imagine then their reaction if you were to tell them that you could construct such an equation, where by using all the inverse operations you used to create the equation, it was impossible to write down the solution using those operations. They probably would have reacted in much the same way we do today when we are told, “If you assume the axiom of choice, you can know about sets which you cannot write down.” This is because this is precisely what an equation not solvable in radicals is. A radical is simply root extraction, the inverse of exponentiation. This takes some getting used to of course, and it took almost 300 years after the solution of the quartic before definitive reasoning for why this happens was given.

Before this could be done, there had to be other, foundational, work done. Work that stretches all the way back to al-Khwarizmı. He gave the solution for an arbitrary real quadratic equation and then proved it geometrically. From here the next logical step is to head for the cubic, but it would take another 500 years before its solution would be accomplished. It was first solved by del Ferro and later by Tartaglia and Cardano. Then the quartic was solved by Ferrari, Cardano’s student, almost immediately after the cubic was solved. Things were looking up, but then, the solutions stopped coming. The quintic equation did not give up its secrets so easily. Viète thought the quintic could be solved by his new notational techniques and said that the purpose of his new methods were “TO LEAVE NO PROBLEM UNSOLVED”. Viète was incorrect in part, however, as were many other quintic solvers. The true story of the quintic would have to wait for Ruffini, who almost proved completely that the quintic was not generally solvable in radicals. This idea was revolutionary, instead of searching for the solution to the quintic, he posed that none existed! While Ruffini’s proof was slightly lacking, Abel’s was not, and in 1824 the problem of the quintic was vanquished. This was not to be the last word on the subject, for Abel would eventually go on to work on solving the quintic in a more general framework than radicals alone. Then Galois would finally place the nail in the coffin with his result that a polynomial is solvable if its Galois group is solvable, and in the process develop modern group theory.

The solvability of the quintic, and polynomials in general, has a rich history spanning centuries and entire fields[*] of mathematics. In this paper, many times references will be made to general solvability as well to other areas of modern mathematics, but the main focus will be on the contributions made by each individual in the line to proving the quintic is not solvable in radicals. We begin in Italy circa 1525.

1 The solution of the Cubic and the Quartic

In the midst of the renaissance there was, in almost every region of Europe, a blooming interest in the arts, culture, science and even mathematics. Particularly in Italy we see the most revered figures of the day, Michelangelo, da Vinci and others. One name more associated with his mathematical contributions rather than his renaissance contributions is Geralamo Cardano. Although he was not the first to solve the cubic equation, he is the one who published it for the public in his *Ars Magna*. The first mathematician credited with solving the cubic equation was Scipione del Ferro. At that time mathematics seemed to be less focused on how much could be know, but how much more you know compared to your adversary. It was in this environment that del Ferro decided to keep his solution a secret[†].

At the end of his life del Ferro relinquished his secrets to his two students Antonio Maria Fiore and Annibale della Neva[1]. Concurrently Niccolo Fontana Tartaglia was working on the same problem. It is suggested [3, 32], [2, 3] that Tartaglia may have plagiarized all or part of del Ferro's solution. While this may or may not be true, one thing that is known is that Tartaglia was able to beat Fiore in a mathematical duel. Therefore Tartaglia must have had a more general solution than Fiore knew. Upon learning that Tartaglia had a solution, Cardano persuaded him to reveal it on the condition that he would not publish it. Cardano kept his word, but then discovered that del

[*] No pun intended.

[†] This began a chain of events which shows just how secrecy among mathematics inevitably leads to trouble. The following is included since one may scarcely talk about the solution of the cubic and its publication without providing adequate context.

Ferro had also discovered a solution. Cardano then decided that the solution should be published, regardless of the deal struck between he and Tartaglia. For one, it was now not Tartaglia's solution he wished to publish, but the solution of the cubic equation. He and Ferrari were given access to del Ferro's notes by della Neva and they confirmed that indeed del Ferro had the same solution as they did. Cardano gave the solution (along with Ferrari's solution of the quartic) in full detail in *Ars Magna*, giving credit to both del Ferro and Tartaglia. Tartaglia was furious and let everyone know about it. Lodovico Ferrari, a former servant and student of Cardano's, challenged Tartaglia to a mathematical duel of their own[4]. At this Tartaglia stalled for as long as he could, until he was offered a position in his home of Brescia which was conditional on his win over Ferrari. The duel was almost a sweep by Ferrari since he knew not only intimately knew the solution of the cubic, he knew how to apply it to the quartic and thus had a more general knowledge of the situation. Things at this point got much worse for Tartaglia, but the entire mathematical community was served by the solution of the cubic and quartic equations regardless of who solved them.

Around fifteen years after *Ars Magna* was published another Italian mathematician, Rafael Bombelli, began to revisit the solution to the cubic. He was a teacher and wanted to enable his students to learn the work of Cardano. He felt that *Ars Magna* was too difficult for students and so he created his own work on the subject. Peculiar to this work are his notation for exponents and the manipulation of quantities that were the square roots of negative numbers. He used a symbolism for exponents that made them obey the same rules as addition when multiplied. That is, instead of a square times a square being a square-square, it would be $x^2 \cdot x^2 = x^4$. He is also the first known person who laid out the rules for working with complex numbers. He used many examples and showed that in some ways they worked like real numbers. Even as he was leading the way for mathematics to come, he himself had trouble accepting the nature of complex numbers. He once said, "the whole matter seems to rest on sophistry rather than on truth [1, 223-224]." From here the complex numbers continue play an important role in the story of the quintic. By providing the field in which the ring of polynomials over that field is algebraically closed, they allow for the completion of the fundamental theorem of algebra. Even more important for our story is the role they play in showing that the quintic is not solvable by radicals.

The mathematics behind the solution to the cubic and quartic is simply an extension of the way the quadratic equation was solved by the ancients. In our modern notation, these solutions are fairly straightforward. The cubic relies on “completing the cube”, but surprisingly the quartic is solved in a very similar manner to the quadratic so there is no need for “completing the square-square”. They build on each other by reducing the given equation to another equation a lesser degree which may be solved by previous methods. The full solution to these equations is given in [3, 36-39]. It is well worth noting that today we can write what took these mathematicians months to work out on 4 pages in a very succinct manner thanks to our notation. In those days just writing a single cubic equation would translate into a paragraph. Imagine what del Ferro, Tartaglia, Cardano and Ferrari could have done if they would have had the advantages of modern notation. Although it was not in their lifetimes, the mathematical community did not have to wait very long for better notation for arithmetical operations.

2 The Role of Symbolism and the Fundamental Theorem of Algebra

It would be hard to imagine the world today existing as it is without the use of algebraic symbolism. Textbooks on mathematics would be sold in volumes and would probably be far more expensive than those we have today. The quadratic formula would still be described in a paragraph and it seems likely that group theory would be unintelligible. In short we use algebraic symbolism so much that we rarely even notice that we are doing so. In his discussion with Meno, Socrates asks the slave boy what the length of the side of a square which had twice the area of a square of side two. The slave boy correctly reasons that the second square would have area 8, but then says 4 would be its side length. Now imagine if the child knew that $\sqrt{x}$ was the length of the side of a square of side 8. He would have been able to immediately given the answer as $\sqrt{8}$ whether he comprehended that 8 was a perfect square or not. The person we have to thank for our modern notation is François Viète. He came up with a system which greatly reduced the amount of space a mathematical statement took up. He thought of this as a side note to solving problems. He discussed the notation in the realm of analysis and believed that he was actually rediscovering the methods of the ancients. In [3, 42] this is described in this way:

> There are some tantalizing examples of this working-backward in Apollonius' seminal work on the conic section, and Viète studied them carefully. He concluded that this analytic art was probably the way the ancients had devised their miraculous proofs: to reach a certain theorem, work backward until you find out what steps are necessary to reach that theorem, then turn around and begin again from that starting point, setting out the steps that you have found to be necessary. Thus, analytic problem solving sets up kind of a scaffolding from which the perfected architecture of the proof can be constructed. At the end, Viète inferred, the ancient masters would remove the scaffolding so the beholder could admire the beauty of the proof, unobstructed by the tools that built it.

Among his ideas is that of the radical symbol $\sqrt{}$ and the arbitrary coefficient. He also considered dimensionality (which he called homogeneous quantities) which was not completely original to him. Now consider describing the very simplistic a_1x in words alone, such is the nature of our symbolism. Viète would write a_1x a little differently than we would. For example, he did not have, or think of, index notation, he also used consonants for known values and vowels for unknowns so a_1x would probably look like BA in Viète's notation. All in all, Viète's contribution to algebra was the transition from numbers to symbols that represented numbers, not only unknown, but known as well. The next notational step would be to move from symbols representing numbers to symbols which represented objects which were not numbers, but could be manipulated in ways similar to numbers. This would lead to modern day algebra, laying the foundations of groups, rings and fields.

Descartes is where we get our modern convention of known quantities being represented by letters close to the beginning of the alphabet and unknowns being represented by those closer to the end. This convention is used so ubiquitously that when someone says

$$\text{Solve:} \quad x^2 + ax + b = 0$$

there is no confusion over what symbol is being solved for. Over the last century, we have moved slightly away from this convention. With the advent of computer algebra systems, which have no sense of context (unless programmed to do so), our definitions must be precise which eliminates the need for convention in most cases. Descartes was also the first one to come up with the basis of analytic geometry and, more important here, he considered aspects of polynomials which would eventually lead to the fundamental theorem of algebra.

Theorem 2.1 (Fundamental Theorem of Algebra). *Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.*

At first glance, the fundamental theorem of algebra almost seems to be at odds with the insolvability of the quintic. On the one hand we are guaranteed solutions exist, on the other hand, we cannot write them down using radicals. It is therefore intriguing that Gauss is the first mathematician credited with believing the quintic was not solvable in radicals since he not only was the first to prove the fundamental theorem of algebra, but proved it in four different ways during his lifetime. The notions of general solutions and solutions in radicals are cleared up by the modern treatment of fields, specifically extension fields and algebraic closures. In fact Gauss says as much, consider this quote from [6] in reference to why Gauss did not attempt a proof of this:

> The other explanation is that he did not attach very much importance to solvability by

radicals. In his thesis referred to above he says: ...“what is called a solution to an equation is, in reality, nothing but the reduction of the equation to prime equations – the solution is not exhibited but symbolized – and if you express a root of the equation $x^n = H$ by $\sqrt[n]{H}$, you have not solved it nor done anything more than if you devise some symbol to denote a root of the equation $x^n + Ax^{n-1} + \ldots = 0$. and place the root equal to this symbol ...”

Gauss is certainly justified saying this since when you attempt to take the nth root of a number which is not a perfect nth power, you have to concede it is irrational and hence there is no perfect numerical representation for it. Therefore extracting the square root is in itself an infinite process and probably should not hold any weight in terms of preference. Now upon consideration of this, we may interpret roots to algebraic equations as simply numbers which satisfy some property (albeit in general a peculiar one) similar to the characteristic of a field. For example if we let α be a root of a particular quintic equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5 = 0$$

then we are merely prescribing a “reduction rule” for α. That is, if we have a particular expression in terms of α and find the above equation we may simply strike it through as zero. It is in this context that Bombelli developed the rules of complex numbers. This is also how the subjects of field extensions and algebraic closure are handled today. Therefore the importance of an equation being solvable in radicals is not that it is the only types of equations we want to work with, but rather its importance is that the roots are made up of elements which are more easily manipulated since they satisfy “simpler” polynomials. This notion of simplicity is built into the names for algebraic numbers which are not expressible in radicals (so-called ultraradicals).

Before the proof of Abel's theorem can be formulated, these ideas have to enter in some context. Here the birth of groups, first as permutations and then in abstract, plays a major role.

3 From Tschirnhausen to Ruffini

Between the time of Cardano and Abel, there were many attempts at solving the quintic. They mostly center around the notion of a resolvent, an equation in terms of roots of the polynomial which takes on certain values upon permutation of the roots, from which Lagrange eventually reveals that the methods of completing the square and cube would not work for the quintic. It is not until Ruffini, though, that serious thought is given to the notion that the quintic may *not* be solvable by radicals. Even then, most mathematicians of the day did not accept this fact, even after they were presented with Ruffini's mostly correct proof. One of the first people to consider a resolvent was Ehrenfried Walter von Tschirnhausen. This account is taken from [6] paraphrased and is probably his translation into modern notation of Tschirnhausen's work. Tschirnhausen begins by considering an arbitrary monic cubic f over the rationals and attempting to find its roots. He then extends the rationals by a primitive cube root of unity, call this field P. Then what remains is to find a field L which contains the splitting field F of the given polynomial. From here Tschirnhausen considers K, an arbitrary extension of P, and then $K(\xi)$ where ξ is a root of f. He then goes on to show that K is degree 2 over P which determines the degree of the resolvent. In general the resolvent of a general polynomial of degree n will have degree $(n-1)!$. Later Etienne Bézout presents a more elementary argument although it is essentially equivalent to Tschirnhausen's.

Euler also wrote on this topic. He conjectured that the roots of a polynomial of degree n were of the form

$$\alpha = \sqrt[n]{A_1} + \cdots + \sqrt[n]{A_{n-1}}$$

which is slightly ambiguous since one may choose any nth root one wishes. To resolve this Euler proposes an alternate form, which for degree 5 is

$$\xi = A_1 \sqrt[5]{A} + A_2 \sqrt[5]{A^2} + A_3 \sqrt[5]{A^3} + A_4 \sqrt[5]{A^4}$$

from which the other roots may be determined by replacing $\sqrt[5]{A^2}$ with $\rho^i \sqrt[5]{A}$ where ρ is a primitive 5th root of unity and i ranges from 1 to 4[6, 3]. Doing this replacement has the same effect as permuting the choice of roots in the resolvent. Euler uses this and shows that there do exist special forms for polynomials of degree 5 which are solvable in radicals. Alexandre-Thophile Vandermonde and Edward Waring also advance in this manner. Ayoub also points out that Waring's knowledge of permutations rivaled that of Lagrange, although he was less systematic and, perhaps as a result, influential.

Lagrange is generally associated with coming to the conclusion that the complexity of the resolvents of these polynomials was sufficient to show that the previous methods of producing formulae for the roots of a polynomial stop at degree 4. He reduced the resolvent from an initial degree 6 in polynomials of degree 4 down to degree 3 by what he calls a "fortuitous circumstance". He applies the same reduction method for degree 5, but can only reduces the resolvent to degree 6, at which point he says that any further reduction "seems to me hardly possible in view of the form of the roots". Later in the same paper he goes on to express his optimism at the prospect of finding new techniques with which to attack the quintic. This optimism Lagrange would carry with him all his life. Almost thirty years later, a new face would emerge and shake up the idea of finding a solution for the quintic equation.

The most important predecessor to Abel in the story of the quintic is Paolo Ruffini. He was the first mathematician to even attempt to prove that the quintic was impossible. Gauss in his doctoral dissertation of 1799 revealed that he had considered the possibility, but as stated earlier, he made no published attempt. This was also the year that Ruffini had published his first proof. Ruffini's proof is complete except for one minor but significant gap. This hole was so elusive, even

Cauchy, one of the only mathematicians who responded favorably to him, did not notice it. The problem was that Ruffini assumed that all algebraic functions can be expressed in terms of rational functions of the roots of the equation. In more modern terminology if L is contained in a radical tower over K, then L/K is itself a radical tower. This was proven in the first step of Abel's two step proof of his theorem. It is unclear whether Ruffini knew that this was necessary for his proof, but it seems that he did not since none of his six proofs contain any heavy discussion regarding it. What is also unclear whether any mathematicians of the day noticed this gap. Assuming its result would make Ruffini's proof acceptable which shows that the community was simply not ready to accept this fact. There were many who responded with concerns, but most were based on lack of understanding rather of Ruffini's arguments rather than noticing the hole in his proof.

Another reason his proofs found little acceptance was that they were "notoriously long and difficult to understand[2]", which may have been a result from his background as a doctor as well as a mathematician. Lagrange's response is recorded in [6] as a cold one, but Ayoub goes on to say that it may have been since Lagrange was old at that time, but he was also considering the first copy of Ruffini's proof which, "is the most intricate and difficult to fathom[6]". It may also be that Lagrange did not want to give his approval since years earlier he had shown that the methods used on previous degrees would not work for the fifth degree, but was still optimistic that new techniques could be found. This revelation that no new methods could be found could be hard to accept after searching your life for a solution to a problem which has none. After he published his first proof, Ruffini responded to the other mathematicians who brought their concerns with altered versions of his proof and in all published six versions of it. Ruffini's original proof and summaries of his other proofs are contained in the often cited and very well presented paper by Ayoub[6]. This account readily shows the ingenuity of Ruffini and that he was aware of (and probably self taught) some elementary group theory which enabled him to prove his conjecture. Basically he showed these resolvents corresponded to subgroups of S_5 and would lead to contradictions to his assumptions on the fields in which they are contained.

Ruffini was well respected through the rest of his life and eventually was elected to the Academy of Sciences. Due to political turmoil in his earlier years, he was in and out of positions at universities, but still had his medical practice. He contracted typhoid fever in 1818 and nearly died, but once his health returned, and in a show of character, he took the opportunity and wrote a paper on contagious typhoid. He eventually would decline a seat at the University of Padua because he wanted to stay close to his patients. Ruffini's work was somewhat lost on the subject of the quintic during the intermediary period and so Abel was not directly influenced by it. Abel did, however, read one of Cauchy's papers that was based on Ruffini's work, so it is possible he was indirectly influenced by Ruffini. Like Columbus discovering the New World for a second time, Abel would go where Ruffini had not, and this time the world would listen.

4 The Resolution of the Problem of the Quintic

The early life of Neils Henrik Abel was one wrought with poverty, due to an economic crisis in Norway at the time[†]. Even so, he was fluent in mathematics and even at this age he already had his mind tuned to the quintic:

> In 1821 Abel entered Royal Fredericks University in Christiania where he continued to excel. During his years at the university Abel believed he had found a general solution to the quintic. However, when pressed for numerical examples by his teachers he soon discovered his solution was not a general one. After this incident Abel set his sights on proving that the quintic was not in fact solvable by radicals[2].

On his next attempt at understanding the quintic, he would not need numerical examples. He proved his result on the quintic while he studied languages in order to travel and meet other math-

[†]For an account of this, see Abel's entry in [4]. It provides a good background for Abel, but until he enters his university studies, little specifically relating to the quintic.

ematicians of the day. He used his own funds to publish his proof in 1824 in a pamphlet that he could take with him on his travels. He was not met with much excitement in many places, but notably August Leopold Crelle took an interest in his work and persuaded Abel to publish a clearer account in his journal (which would come to be referred to simply as "Crelle") in 1826. Later in his professional life, he had more success being recognized, indeed:

> Such mathematicians as Gauss and Legendre sang his praises. Unfortunately, Abel was unable to secure a university position even with such mathematicians trying to find him work[2].

He did not continue down the path of solvability consistently although he eventually knew something equivalent to Evariste Galois's later result[4]. Instead he focused on competing with Carl Gustav Jacob Jacobi in the area of elliptic functions. His health began to deteriorate as he continued searching for positions at universities. On April 6th, 1829 he died, and then he finally got his well-deserved seat at a university:

> On the 8th of April Crelle sent word that he had finally found a position for Abel so he would live in poverty no longer[2].

It is amazing to me the work that Abel did at such a young age. He completely solved a problem which had been open for almost three centuries. In his proof, he breaks the problem into two steps, the first being the step that Ruffini neglected to include in his proof, and the second was the proof along the same lines as what Ruffini had produced. The first step was to show that all algebraic functions can be expressed in terms of rational functions of the roots of the equation, that is if L is contained in a radical tower over K, then L/K is also a radical tower. Its proof depends on the following well-known result, as well as the others following which are taken from and proven in [5]. I include the proof of this theorem since[†] it is the gap in Ruffini's proof and is one of Abel's major contributions.

[†] And it is a very nice proof!

Lemma 4.1. *Let F be a field containing a primitive qth root of unity. If $a \in F^*$ is not a qth power, then $x^q - a$ is irreducible.*

Lemma 4.2. *Assume that $x^q - a \in F[x]$ is irreducible and that α is a root. Let γ be an element of $F(\alpha)$ with $\gamma \notin F$. Then there is a $\beta \in F(\alpha)$ such that $\beta^q \in F$ and*

$$\gamma = b_0 + \beta + b_2\beta^2 + \cdots + b_{q-1}\beta^{q-1}$$

where $b_0, b_2, \ldots, b_{q-1} \in F$.

Lemma 4.3. *Let q be a prime, and ζ a primitive qth root of unity. Then, for each integer i,*

$$1 + \zeta^i + \zeta^{2i} + \cdots + \zeta^{(q-1)i} = \begin{cases} 0 & \text{if } q \text{ does not divide } i, \\ q & \text{if } q \text{ divides } i, \end{cases}$$

Lemma 4.4. *Consider the extension L/K. Let $y \in L$. Then the irreducible polynomial for y over K splits into linear factors in $L[x]$.*

This is the crux[5] for the proof of Theorem 4.6.

Lemma 4.5. *Let E/K be an extension field, q a prime, and $a \in E$ an element such that $x^q - a \in E[x]$ is irreducible. Let α be a root of $x^1 - a = 0$. Set $M = E(\alpha) \cap L$ and $M_0 = E \cap L$. If $M \neq M_0$, then M/M_0 is a radical extension. More precisely, there is a $\beta \in M$ such that $\beta^q \in M_0$ and β generates M over M_0.*

Theorem 4.6. *If L/K is contained in a radical tower, then L/K is itself a radical tower.*

Note that $\subset$ is proper containment.

Proof: Suppose that E/K is a radical tower and that $L \subseteq E$. We have

$$K = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$$

where $E_{i+1} = E_i\left(\sqrt[q_i]{a_i}\right)$, q_i being a prime and $a_i \in E_i$. Now consider the tower

$$K = E_0 \cap L \subseteq E_1 \cap L \subseteq \cdots \subseteq E_{m-1} \cap L \subseteq L. \tag{1}$$

If $E_{i+1} \cap L = E_i \cap L$ there is nothing that need be said. If $E_{i+1} \cap L \neq E_i \cap L$ then Lemma 4.5 shows that $E_{i+1} \cap L/E_i \cap L$ is a radical extension (of degree q_i). Thus, after eliminating equalities, equation 1 demonstrates L as a radical tower over K. Q.E.D.

With this quite elegant proof, step 1 of Abel's theorem is complete. As with Ruffini's proof, step 2 is proved using cases relating to the subgroups of S_5. How is it that Abel could come up with such a ground breaking result and have nowhere to go for work? It is very unusual indeed, but given the brief time he was here, Abel still contributed masterfully to the body of mathematics. This time everyone would know that the quintic was genuinely unable to be solved by radicals.

Reflections

The road to demonstrating fully that the quintic is not solvable in radicals, and later that of showing which polynomials are solvable, was a rough one for those involved. Nearly everyone who touched this subject had a rough life. It could very well be called Curse of the Quintic[†]

Tartaglia lost his prestige at the hands of Ferrari and never fully recovered professionally.

Cardano published the solution, lived a long unhappy life. His only son was executed for murder; he was later put on trial by the Inquisition for attempting to cast the horoscope of Christ.

Ferrari poisoned, probably by his sister, over an inheritance dispute.

Ruffini contracted typhoid and his proof was never fully accepted, although he seems to have fared better than the others.

[†]This is an expanded and corrected version of "The curse of the cubic" located at http://math.bu.edu/INDIVIDUAL/jeffs/mathematicians.html

Abel died at age twenty-nine, days after he received a position which would have kept him out of poverty.

Galois died young at twenty in a duel for a reason that may never be known, after spending time in jail for his politics.

As with any worthy problem in mathematics, this one spawned new areas of mathematics at almost every level. Some topics created out of the search to solve the quintic are complex numbers, elliptic functions, groups and Galois theory. Now these disciplines have their own monumental questions which may have never been asked had no one searched for a solution to polynomials of ever higher degree. We now know that we need some extra machinery in order to handle these, but they are not out of our reach. Will there ever be problems which we cannot solve? Gödel showed us that no matter what axioms we choose, there will be problems we will not be able to decide with them. That does not necessarily mean that they have no solution, simply we have yet to develop a proper system in which to handle them. This song certainly sounds familiar to me, and although the melody is very complicated (but at the same time elegant and beautiful) the lyrics are clear, they are those of Viète, and they remind me that we will always be striving, no matter what the difficulty, to

LEAVE NO PROBLEM UNSOLVED

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